Name: _____

- This packet is due at **12:00 pm PDT (noon) on Friday, April 16**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Use the sequence of partial sums to determine whether the series converges or diverges.

(a) $\sum_{n=0}^{\infty} \frac{n}{2}$

$S_0 = \frac{0}{2}$

$S_1 = \frac{0}{2} + \frac{1}{2} = \frac{1}{2}$

$S_2 = \frac{0}{2} + \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$S_3 = \frac{0}{2} + \frac{1}{2} + \frac{2}{2} + \frac{3}{2} = \frac{6}{2}$

$\vdots$

$S_k = \frac{0}{2} + \frac{1}{2} + \frac{2}{2} + \cdots + \frac{k}{2} = \frac{1}{2}(1 + 2 + \cdots + k) = \frac{1}{2} \cdot \frac{k(k+1)}{2}$

$\sum_{n=0}^{\infty} \frac{n}{2} = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \frac{k(k+1)}{4} = \infty$

The series diverges

(b) $\sum_{n=1}^{\infty} (1 - (-1)^n)$

$S_1 = 1 - (-1) = 2$

$S_2 = \underbrace{1-(-1)}_{2} + \underbrace{(1-(-1)^2)}_{0} = 2$

$S_3 = \underbrace{1-(-1)}_{2} + \underbrace{(1-(-1)^2)}_{0} + \underbrace{(1-(-1)^3)}_{2} = 4$

$S_4 = \underbrace{1-(-1)}_{2} + \underbrace{(1-(-1)^2)}_{0} + \underbrace{(1-(-1)^3)}_{2} + \underbrace{(1-(-1)^4)}_{0} = 4$

$\vdots$

$$S_k = \begin{cases} 2k & k \text{ odd} \\ 2(k-1) & k \text{ even} \end{cases}$$

$\sum_{n=1}^{\infty} 1 - (-1)^n = \lim_{k \to \infty} S_k = \infty$

2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$

$a_n = 4 \cdot \left(\frac{3}{4}\right)^n$ for $n \geq 0$

$\sum_{n=0}^{\infty} 4 \cdot \left(\frac{3}{4}\right)^n = 4 \cdot \frac{1}{1 - \frac{3}{4}} = \boxed{16}$

(b) $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{\pi^n}{3^n} \cdot \frac{1}{3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n = \infty$

because $\frac{\pi}{3} > 1$

(c) $10 - 2 + 0.4 - 0.08 + \cdots$

$a_n = 10 \cdot \left(-\frac{1}{5}\right)^n$ for $n \geq 0$

$\sum_{n=0}^{\infty} 10 \cdot \left(-\frac{1}{5}\right)^n = 10 \cdot \frac{1}{1 + \frac{1}{5}} = 10 \cdot \frac{5}{6} = \frac{25}{3}$

(d) $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n} = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{\sqrt{2} - 1}}$

3. Use a geometric series to demonstrate that $0.\overline{9} = 0.9999999\ldots$ is equal to 1.

$$0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

$$= \frac{9}{10}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$$

$$= \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}}$$

$$= \frac{9}{10} \cdot \frac{10}{9}$$

$$= 1$$

4. Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

(a) $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

① $\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} \rightarrow 2 = A(n+1) + B(n-1)$

$n=1 \rightarrow 2 = A(2) \rightarrow A = 1$

$n=-1 \rightarrow 2 = B(-2) \rightarrow B = -1$

② $\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$

$\rightarrow S_2 = \frac{1}{2-1} - \frac{1}{2+1}$

$S_3 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

$S_4 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$

$S_5 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}$

$\vdots$

$S_k = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{k-2} - \frac{1}{k} + \frac{1}{k-1} - \frac{1}{k+1}$

$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} = \boxed{\frac{3}{2}}$

b) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$

$S_1 = \ln(1) - \ln(2)$

$S_2 = \ln(1) - \ln(2) + \ln(2) - \ln(3)$

$S_3 = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4)$

$\vdots$

$S_k = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4) + \cdots + \ln(k) - \ln(k+1)$

$= \ln(1) - \ln(k+1)$

$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{k\to\infty} \ln(1) - \ln(k+1) = \boxed{-\infty}$

5. For each of the following series, apply the divergence test. State whether or not the divergence test allows you to make a conclusion about the convergence of the series. You don't have to compute any of the sums; just tell me the result of applying the divergence test.

(a) $\sum_{n=2}^{\infty} \frac{2n}{n^2-1}$

$\lim_{n\to\infty} \frac{2n}{n^2-1} = 0$ (since the degree of the numerator is less than the degree of the denominator)

→ No conclusion

(b) $\sum_{n=0}^{\infty} \frac{(3n-2)(n+7)}{n^2+2n-1}$

$\lim_{n\to\infty} \frac{(3n-2)(n+7)}{n^2+2n-1} = \lim_{n\to\infty} \frac{3n^2+19n-14}{n^2+2n-1} = 3$

→ The series diverges

(c) $\sum_{n=0}^{\infty} \frac{2^n}{3^{n/2}}$ $\lim_{n\to\infty} \frac{2^n}{3^{n/2}} = \lim_{n\to\infty} \left(\frac{2}{3^{1/2}}\right)^n = 0$ (since $\frac{2}{\sqrt{3}} < 1$)

→ No conclusion

(d) $\sum_{n=0}^{\infty} e^{-n^3}$ $\lim_{n\to\infty} e^{-n^3} = 0$

→ no conclusion

(e) $\sum_{n=1}^{\infty} e^{-5/n}$

$\lim_{n\to\infty} e^{-5/n} = 1$

→ The series diverges

(f) $\sum_{n=0}^{\infty} \sin(n)$

$\lim_{n\to\infty} \sin(n)$ DNE

→ The series diverges

(g) $\sum_{n=1}^{\infty} \frac{1-\cos(2/n)}{\sin(1/n)}$

$\lim_{n\to\infty} \frac{1-\cos\left(\frac{2}{n}\right)}{\sin\left(\frac{1}{n}\right)}$ ↓ apply L'Hôpital ($\frac{0}{0}$ case)

$= \lim_{n\to\infty} \frac{\sin\left(\frac{2}{n}\right)\cdot\left(-2n^{-2}\right)}{\cos\left(\frac{1}{n}\right)\cdot\left(-n^{-2}\right)}$

$= \lim_{n\to\infty} \frac{2\sin\left(\frac{2}{n}\right)}{\cos\left(\frac{1}{n}\right)} = 0$

→ no conclusion

(h) $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{(\ln(n))^2}{\sqrt{n}}$ apply L'Hôpital ($\frac{\infty}{\infty}$ case)

$= \lim_{n\to\infty} \frac{2\ln(n)\cdot\frac{1}{n}}{\frac{1}{2}n^{-1/2}}$

$= \lim_{n\to\infty} \frac{4\ln(n)}{n^{1/2}}$

$= \lim_{n\to\infty} \frac{4\left(\frac{1}{n}\right)}{\frac{1}{2}n^{-1/2}}$

$= \lim_{n\to\infty} \frac{8}{n^{1/2}} = 0$

→ no conclusion

6. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$

$\lim_{k\to\infty} \frac{k^2}{k^2-1} = 1$

By the divergence test, the series diverges

(b) $\sum_{n=1}^{\infty} \cos(\frac{1}{n})$

$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos(0) = 1$

By the divergence test, the series diverges

(c) $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

$= \frac{1}{3}\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} + \frac{2}{3}\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$

$= \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n} + \frac{2}{3}\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n-1}$

$= \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right) + \frac{2}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}\left(\frac{3}{2}\right) + \frac{2}{3}(3)$

$= \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$

(d) $\sum_{k=1}^{\infty} (\cos(1))^k$

$0 < \cos(1) < 1 \rightarrow \sum_{k=1}^{\infty} (\cos(1))^k = \cos(1)\sum_{k=1}^{\infty} \cos(1)^{k-1}$

$= \cos(1)\cdot\sum_{k=0}^{\infty} \cos(1)^k$

$= \cos(1)\cdot\frac{1}{1-\cos(1)}$